Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-11-29

Warm-up

Question 1. Let C be a clockwise closed curve in the plane which encloses some region D. Which of the following integrals computes the area of D? (What are the values of the other integrals?)

$$\int_C x\,\mathrm{d}x, \quad \int_C (x\,\mathrm{d}y - y\,\mathrm{d}x), \quad \int_C y\,\mathrm{d}x$$

Question 2. Show that the angle formed between the xy-plane and any tangent plane of the parametric surface $\mathbf{r}(z,\theta) = \langle z\cos\theta, z\sin\theta, z\rangle$ is always the same, and find this angle. Also identify what surface this is.

Question 3. Suppose that $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ that is always parallel to the xy-plane, i.e. it has zero z-component. Does it follow that $\nabla \times \mathbf{F}$ is vertical at all points?

Computations

These problems are taken from last year's final, with slight adjustments.

Problem 1. Let $P(t), Q(t), R(t)$ be single-variable functions. Define $f(x,y,z)$ as

$$f(x,y,z) = \int_0^x P(t)\,\mathrm{d}t + \int_0^y Q(t)\,\mathrm{d}t + \int_0^z R(t)\,\mathrm{d}t.$$

Let $\mathbf{F} = \nabla f$.

(a) Compute $\mathbf{F}$ in terms of P, Q, R.

(b) Let S be the hemisphere $x^2+y^2+z^2=1, z \geq 0$, oriented upwards. Let ∂S be the oriented boundary of S. Compute

$$\int_{\partial S} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{r}.$$

Problem 2. Let $\mathbf{F}$ be the vector field $\langle x - \frac{2}{3}x^3, -\frac{4}{3}y^3, -\frac{8}{3}z^3\rangle$. Find the closed surface S in $\mathbb{R}^3$ which maximizes the value of the (outwards) flux integral

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Problem 3. Suppose that S is a closed surface parametrized by s, t over the region $0 \leq t \leq 1$, $0 \leq s \leq 2$. Show that the volume enclosed by S is

$$\left|\frac{1}{3}\int_0^1\int_0^2 \left(x\left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right) + y\left(\frac{\partial x}{\partial t}\frac{\partial z}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial z}{\partial t}\right) + z\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)\right)\mathrm{d}s\,\mathrm{d}t\right|$$

(where the integrand is understood to be written in terms of s, t).

Problem 4. Let Γ be the parametrized polar curve $r = t, \theta = 8\pi t, 0 \leq t \leq 1$.

(a) Rewrite Γ as an ordinary parametric curve $x = f(t), y = g(t)$.

(b) Compute $\int_\Gamma \langle x, y\rangle \cdot \mathrm{d}\mathbf{r}$.